QUANTITATIVE COMPARISON OF LDA- AND PIV-MEASUREMENTS

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Abstract

Particle Image Velocimetry (PIV) is a well established measurement technique for the qualitative as well as quantitative description of flow fields and structures. To achieve a quantitative estimation of the PIV images analyzed with different algorithms, extensive investigations on mostly synthetic images have been published in the past (Raffel et al (1998), Keane & Adrian (1992)). For example, Willert & Gharib (1991) showed that the error based on real measured PIV images is one order of magnitude larger when compared with theoretical estimated errors based on synthetically generated images where the flow field is a priori known. To measure the turbulent velocity fluctuations, however, Laser Doppler Anemometry (LDA) is still the most widely accepted non intrusive optical method. LDA measurements of turbulent quantities have also been validated extensively against other measurement techniques such as hot wire measurements.
The experiments presented in this article were conducted for a quantitative validation of PIV. Combined LDA and PIV measurements were carried out in the wake of a circular cylinder such that both measurements were taken simultaneously at the same position and during the same time with the same number of samples. The PIV raw images were subsequently analyzed with various commercial codes. Additionally, the Optical Flow was used as a new analysis method and compared with the classical algorithms.
The analysis' show that at least 2500 samples are needed to reach less than 1% deviation to the value at a statistic of 4000 samples. The velocity fluctuation at 2500 samples differ about 3% from the converged value. Regardless, the difference between LDA- and PIV-measurements is still huge, and varies strong with changing the PIV processing parameters.

Introduction

PIV is a non intrusive laser-optical principle to measure planar two- or three-component velocity fields. The base method of determining the flow velocity based on an observation of the shift of tracer particles in a fluid during a defined time interval has been known for a long time. First developments in the direction of PIV were published by Barker & Fourney (1977), Dudderar & Simpkins (1977) and Grousson & Mallick (1977) in conjunction with Laser

Speckle Velocimetry. Adrian (1984) and Pickering & Halliwell (1984) suggested to observe the shift of an ensemble of particles, and this led to the development of PIV technology. In contrast to the first experiments, modern PIV systems now benefit from the advantages of CCD and CMOS cameras, up to date laser technology and powerful computers. Combined with extensive programs for digital image processing, modern PIV is known as a very powerful measuring principle.

Nevertheless, the uncertainty of real PIV measurements is difficult to quantify, and the measurement uncertainty usually depends strongly on the analysis chain actually used (preprocessing, velocity calculation, postprocessing) as well as the choice of the analysis parameters. In particular, the number of PIV velocity fields used to derive averaged quantities in addition to the use of noise reduction algorithms and the choice of the interrogation area size all influence both the magnitude of the Reynolds stresses as well as the mean quantities.

New image processing algorithms like the Optical Flow, introduced in Horn & Schunk (1981), improved by Rhunau (2005) and further adapted by Kapulla et al (2010), are expected to deliver better results for special applications.

To quantify the uncertainty of the mean velocity and the turbulent velocity fluctuations, PIV and LDA measurements were simultaneously performed at the same position, during the same time and with the same number of samples, with the LDA-measured velocities used as reference.

Methods

For the LDA measurements, a two-component DANTEC LDA System consisting of an Argon-Ion Laser and two Burst Spectrum Analyzers was used. The bursts were analyzed with the software BSAFlow 2.0.0. To synchronize the LDA bursts with the PIV images the trigger signals from the PIV-Q-Switches were logged and written into the LDA files. Three methods were chosen for the calculation of the mean velocity and the velocity fluctuations of the LDA recordings. In method one all of the approximately 100 000 bursts were analyzed. Method two used only the bursts between the first and the last Q-Switch-Triggers, which resulted in about 80 000 samples. In method three only the first burst (following Q-Switch-1-Trigger) was used, resulting in 4096 samples. Method 3 guarantees the same number of samples and the same statistic for LDA and PIV.

The PIV measurements were recorded at a repetition rate of 15 Hz, resulting in a measuring time of 4 minutes and 33 seconds for 4096 double frame single exposure images. Since the analysis of the PIV images is the main focus of the present study, a number of different algorithms implemented in various commercially available software codes were used without knowing the results of the LDA-measurements. For the calculation of the mean and turbulent quantities only the validated and non rejected vectors were used.

The basic analysis was carried out with FlowManager 3.7 (Dantec Dynamics). For comparison with this analysis method DaVis 7.2.2.260 (LaVision) was used, as well as the Optical Flow as implemented at Ism and already published in Kapulla et. al. (2009). The algorithms used by FlowManager are published in Dantec (2002). In all cross-correlation methods the extended adaptive correlation was used with validation algorithms to reject non physical outliers. For further processing, the outlying vectors have not been taken into account. The different analysis parameters for image processing with the cross-correlation technique are shown in Table 1.

Tab. 1: Settings of the correlation algorithms

Software	FlowManager Run 1	FlowManager Run 2	FlowManager Run 3	FlowManager Run 4	DaVis
Image pre-processing	subtract mean	subtract mean	subtract mean	subtract mean	Intensity normalization
Interrogation Area	128x128 64x64 32x32	128x128 64x64 32x32	128x128 64x64 32x32	128x128 64x64 32x32	64x64 32x32 16x16
Overlap	0%	0%	0%	0%	50%
Refinement Steps	1,2,1	9,9,9	9,9,9	1,2,1	2,2,2
Peak Validation	2.0	2.0	2.0	1.0	1.3
Subpixel Refinement	Yes	Yes	No	Yes	Yes
Deforming Windows	Yes	Yes	Yes	Yes	No
Local Neighbourhood validation	No	No	No	No	Yes

Another promising technique to analyze PIV double frame images is the optical flow method. The first description of this method was given by Horn & Schunk (1981), in which the authors show the potential of the method to calculate dense motion fields. In contrast to the standard 2D FFT based correlation methods, optical flow treats the whole velocity field. In this method, the motion of the particles between a double image recording is taken as a continuous system of flow structures. With the assumption of a continuous system, a brightness transport equation can be derived to calculate the velocity field for every pixel in a single step.
To derive the governing equations it is assumed that the position dependent image brightness I(x,y,t) is conserved in time, which - using the Einstein summation convention - leads to:

$$G(t) = \int_{\Omega(t)} I(x,y,t)\,d\Omega \quad \text{and} \quad \frac{dG}{dt} = \int_{\Omega} \left(\frac{\partial I}{\partial t} + \frac{\partial I}{\partial x_i} \cdot u_i \right) d\Omega = 0$$

The unknown velocity components u_i denoted as u and v for the two dimensional cases considered here describe the apparent motion of the brightness pattern recorded. With this equation it is not possible to calculate the movement of the brightness pattern along iso-brightness contours at right angles to the brightness gradient. To overcome this shortcoming, an additional constraint must be introduced. According to the idea of Horn & Schunck (1981) a simple first order regularization term is used. With the assumption that neighboring pixels in the image represent similar velocities, the magnitude of the optical flow velocity gradient ε_R has to be bounded :

$$\varepsilon_R = \frac{\partial u_i}{\partial x_k} \cdot \frac{\partial u_i}{\partial x_k}$$

To obtain a unique solution for the velocities u and v we must integrate the additional constraint into the conservation equation using a Lagrange multiplier α:

$$F = \int_{\Omega} \left[\left(\frac{\partial I}{\partial t} + \frac{\partial I}{\partial x_i} \cdot u_i \right) \cdot \left(\frac{\partial I}{\partial t} + \frac{\partial I}{\partial x_j} \cdot u_j \right) + \alpha^2 \cdot \left(\frac{\partial u_i}{\partial x_k} \cdot \frac{\partial u_i}{\partial x_k} \right) \right] d\Omega \rightarrow \text{Min}$$

The corresponding Euler Lagrange differential equation is derived from the functional F by use of variational calculus.

$$\frac{\partial I}{\partial x_i} \cdot \frac{\partial I}{\partial x_j} \cdot u_j - \alpha^2 \cdot \frac{\partial^2 u_i}{\partial x_j^2} = -\frac{\partial I}{\partial x_i} \cdot \frac{\partial I}{\partial t} \text{ in } \Omega$$

$$n_i \cdot u_i = 0 \text{ on } S_{\Omega}$$

This system of differential equations may be solved numerically using standard methods such as FDM (finite difference method) published by Mitiche & Mansouri (2004) or FEM (finite element method), shown in Ruhnau et al (2004). It is also possible to use higher order regularization terms as presented by Corpetti et al (2006) which use more physical background than the original model of Horn & Schunck. It should be noted, however, that this implies an increased numerical effort. The major problem of all optical flow schemes is the estimation of large displacements as discussed in Heitz et al (2007). To overcome this problem, the solution procedure is embedded in a multi resolution and multi scale scheme similar to Ruhnau et al (2004) and Kapulla et al (2009).

Experimental Setup

The measurements were conducted in the wake of an infinitely circular cylinder at different positions. For the generation of the flow, the Göttingen type wind tunnel (open area 0.65 x 0.65 m²) at the University of Rostock was used. The LDA was aligned perpendicular to the incident velocity so as to measure the velocity components u and v. Due to the expanded field of view of the PIV-camera, this alignment allowed simultaneous measurements at the same place. Figure 1 shows a principle sketch of the setup.

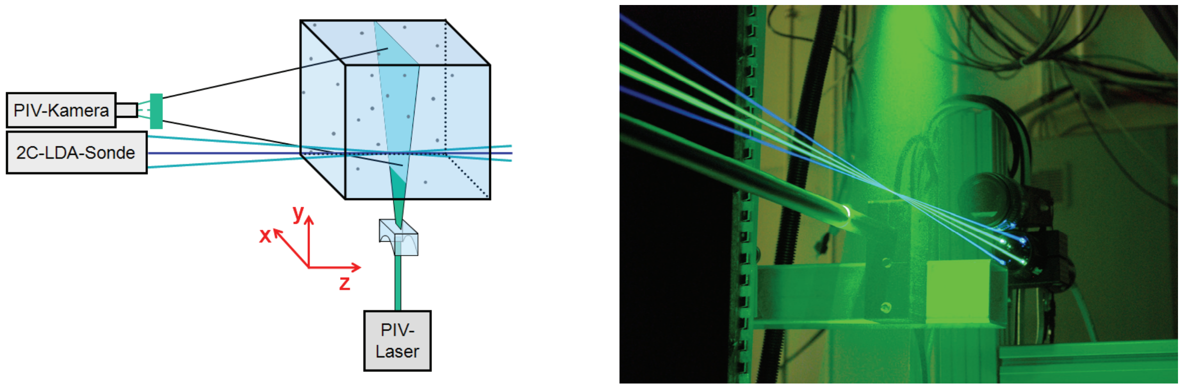
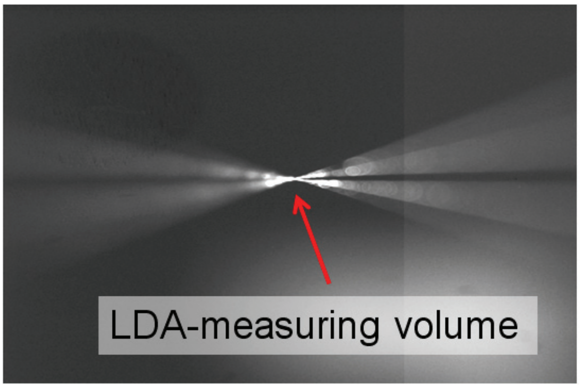
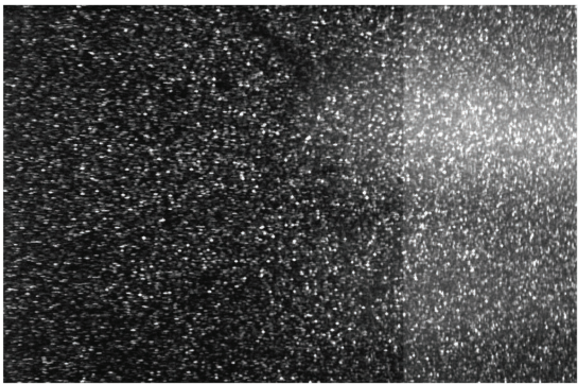


Figure 1: Principle setup of simultaneous PIV/LDA-measurements.

To distinguish the LDA from the PIV signal and also to enable undisturbed cross-talk-free PIV measurements, both systems had to be separated optically. The LDA measurements were therefore performed with a 2-component-2-beam-LDA in backscatter mode, equipped with an Argon-Ion-laser using the wavelengths λ_1 = 514.5 nm and λ_2 = 488 nm. The PIV measurements were carried out with a frequency-doubled Nd:YAG-laser at a wavelength of λ_3 = 532 nm. The optical separation of both signals was realized with a narrow band interference filter of 3 nm bandwidth in front of the PIV-camera. With the optical configuration described above, it was possible to perform simultaneous LDA and PIV measurements at the same location.



a. with simple color filter, 10 nm bandwidth



b. with interference filter, 3 nm bandwidth

Figure 2: LDA beams recorded with the PIV camera using a simple color filter in front of the lens (a) and the same recording with narrow band interference filters in front of the PIV camera (b).

The effect of the narrow band interference filter in front of the PIV camera is shown in Figure 2. After removing the interference filter, it was possible to precisely locate the LDA measuring volume within the PIV light sheet.

PIV and LDA results

For a first comparison of the PIV and LDA results, measurement locations at low and elevated rates of turbulence fluctuations in the flow past the cylinder were chosen. The mean of the two measured velocity components u and v, the corresponding autocorrelation of their fluctuations $\overline{u'u'}$ and $\overline{v'v'}$, the number n of averaged vectors and the deviation of the computed quantities related to the LDA measurements (u*, v*, k*) for the high turbulence case are presented in Table 2, and Table 3 for the low turbulence case. In contrast to the usual definition using all three components of the velocity vector, here we have calculated the turbulent kinetic energy k according to

$$k = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'}\right).$$

The FlowManager processing setup with the Refinement steps 1, 2, 1 (see Table 1, second column) shows the best results in comparison with LDA. These best run results of the FlowManager analysis are presented in Tables 2 and 3. For the high turbulence case it was found that the PIV-based mean velocity deviates by approximately 7% for the high velocity value u when compared with the LDA result. This velocity, u≈11 m/s, corresponds to a pixel displacement of approximately 8. For the low velocity value v with an average of less than one pixel shift, but also a high velocity fluctuation corresponding to an instantaneous shift of approximately 8 pixels, the deviation is greater than 40%. In contrast to this, it is solely the Optical Flow analysis which shows small deviations also for small pixel shifts. The deviations for both velocity components calculated with the optical flow method have the same order of magnitude $\varepsilon \approx 8$ %, irrespective of the underlying pixel shift. The turbulent quantities ($\overline{u'u'}$, $\overline{v'v'}$ and k), measured with PIV are consistently smaller than those measured with LDA. This appears to be a principle problem of PIV and the optical detection of particles with cameras.

Tab. 2: Results from the different analysis runs for high turbulence case

Principle	LDA	PIV/FlowManager	PIV/DaVis	PIV/Optical Flow
u [m/s]	11.5	10.7	10.8	10.6
v [m/s]	0.7	1.0	1.0	0.6
$\overline{u'u'}$ [m²/s²]	40.79	38.59	44.7	40.33
$\overline{v'v'}$ [m²/s²]	221.55	192.70	180.2	175.95
n	4096	3060	3922	4096
100-{(u*-u)/u*} [%]	100	93	94	92
100-{(v*-v)/v*} [%]	100	145	143	92
k/k* [%]	100	88	86	82

For the low turbulence case (Table 3) the calculated high velocity values u≈24.7 m/s are more consistent throughout the different analysis algorithms than for the high turbulence case. Excluding the optical flow based result, the low velocity value v≈0.3 m/s results deviate from the nominal LDA value by 30 to 50 %; this matches the magnitude of the high velocity case (30 %). Due to the very low turbulence, the absolute turbulent kinetic energy is very small and the relative deviation therefore becomes very high.

Tab. 3: Results from the different analysis runs for low turbulence case.

Principle	LDA	PIV/FlowManager	PIV/DaVis	PIV/Optical Flow
u [m/s]	24.6	24.9	24.8	24.7
v [m/s]	0.4	0.3	0.2	0.1
$\overline{u'u'}$ [m²/s²]	0.12	0.08	0.5	1.25
$\overline{v'v'}$ [m²/s²]	0.10	0.10	0.8	0.89
N	4096	3688	4048	4096
100-{(u*-u)/u*} [%]	100	101	101	101
100-{(v*-v)/v*} [%]	100	65	40	26
k/k* [%]	100	84	587	869

When using ≈4000 samples, the results for the low and high turbulence cases as presented in Tables 2 and 3 for the mean velocities may be summarized as follows: for the larger velocity component, u, in the main flow direction, a maximum uncertainty of 7 % is found. For considerably smaller velocity component v, deviations in the range of 30 - 50 % are seen. These deviations depend on the cross-correlation based software package used. The turbulent quantities for the high turbulence case are measured with approximately 20% uncertainty in comparison with the LDA results, whereas the relative deviation at low turbulence is very large due to low absolute values. This result may be affected by randomly distributed correlation maxima due to small particle shifts.

Convergence plots are then used for the mean and fluctuating quantities, and the results are normalized to the algorithm individual final value for the maximum number of samples available. This leads to all of the plots converging to one, and the differences of the results as discussed above are neglected, Figure 3. It is found that for the mean velocities at least 2500 samples are necessary to achieve results which deviate less than 1% from the more accurate value for n≈4000. In line with expectations, the turbulent quantities reach 5% of the converged value at around 500 samples. To attain less than 1% deviation from the converged values at n≈4000, 3000 samples is not enough. Furthermore, it may be noted that LDA converges slightly faster, but not significantly.

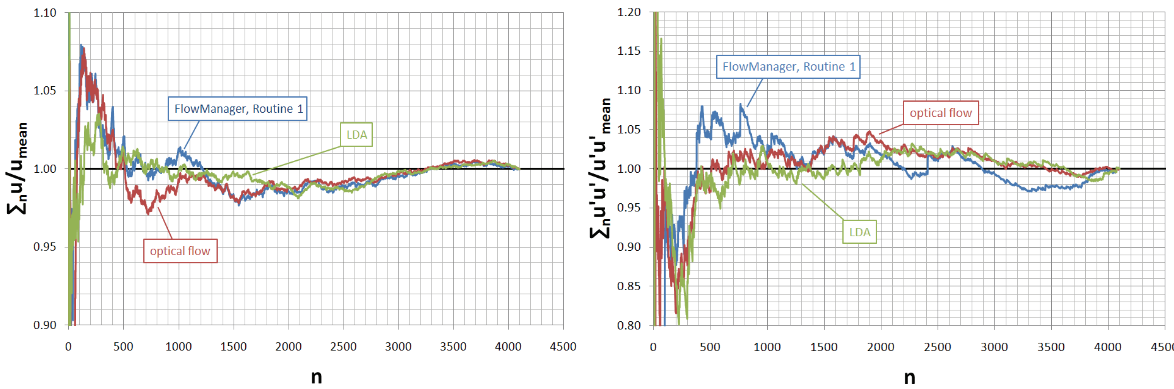


Figure 3: Convergence of the velocity component u (left) and the fluctuation u'u' (right) for high turbulence case

To consider also the absolute deviations for the different analysis algorithms applied, we present convergence plots at the high turbulence case where the results are related to the LDA reference case, Figure 4. This comparison was performed for the cross-correlation based FlowManager result in the left part of Figure 4, and the optical flow algorithm in the right part of Figure 4. One finds that the absolute difference between the FlowManager and LDA based result as well as between the Optical Flow and LDA based result are both quite small. The velocity component u shows a difference of about 1 m/s for both analysis procedures. This corresponds to a relative deviation of about 10% as already stated above. The absolute difference in v converges to close to zero because of the low absolute values. However, the relative deviation constitutes more than 20% (see Tables 2 and 3). The result for $\sqrt{u'u'}$ also shows small absolute differences, whereas for $\sqrt{v'v'}$ we obtain differences of more than 1 m/s.

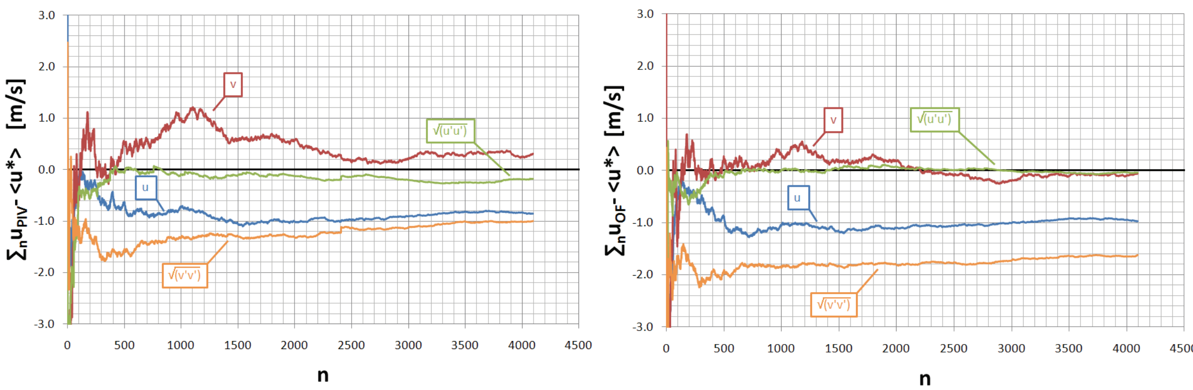


Figure 4: Convergence of the difference of the velocity components and the fluctuations to the converged LDA results for Cross Correlation with FlowManager (left) and Optical Flow (right) at the high turbulence case.

Taking the mean of the relative deviation of 7 different measurements and the 5 processing setups (optical flow, DaVis and 3 times FlowManager), the deviation of the higher velocity values about 6 % whereas deviation of the lower velocity values more than 55 %. With a relative error of more than 300 % compared with the LDA measurements for the case of low turbulence, the calculated fluctuations are far from being regarded as satisfying. For the high turbulence case, both cross correlation and Optical Flow reach only 80 % of the turbulent fluctuations measured with LDA. However, the absolute deviation over all is negligible. This indicates that a quantification of the turbulence level with PIV, especially the extraction of maximum values, must be done very carefully. Furthermore, the quantification of high turbulence seems to depend strongly on the post processing algorithm. Additionally, the results show the tendency of cross correlation based algorithms to underestimate the velocity fluctuations in high turbulence areas.

Parameter Dependence

All PIV images were processed with FlowManager using different user adjustable parameters, Table 1 Runs 1 through 4. In particular the number of refinement steps was varied, and the peak validation ratio and subpixel refinement option (by Dantec) was switched on and off. The results of these analyses' runs for the high turbulence case are presented in Table 4. Somewhat surprisingly we found that increasing the number of refinement steps does not necessarily lead to more accurate results (Run 1 versus Run 2), but the number of valid vectors increases while the peak validation ratio is kept constant at 2.0. The use of the subpixel refinement option results in better values for the fluctuating quantities (Run 2 versus Run 3).

Tab. 4: Results for high turbulence of the different FlowManager settings

Software	LDA	FlowManager Run 1	FlowManager Run 2	FlowManager Run 3	FlowManager Run 4
Refinement Steps	-	1,2,1	9,9,9	9,9,9	1,2,1
Peak Validation	-	2.0	2.0	2.0	1.0
Subpixel Refinement	-	Yes	Yes	No	Yes
u [m/s]	11.5	10.7	10.5	10.7	10.3
v [m/s]	0.7	1.0	1.0	1.1	0.9
u'u' [m²/s²]	40.79	38.59	44.15	38.26	59.55
v'v' [m²/s²]	221.55	192.70	190.73	192.19	198.13
N	4096	3060	3232	3005	4090
100-{(u*-u)/u*} [%]	100	93	91	93	90
100-{(v*-v)/v*} [%]	100	145	147	154	128
k/k* [%]	100	88	90	88	98

The strongest impact on the results of the turbulent quantities is noted for the peak validation ratio (Run 1 versus Run 4). A decreased peak validation ratio results in a considerable number of false positive vectors. These false positives may then be conceptually treated as the addition of noise with the consequence of higher turbulent quantities calculated. If we assume the false positive vectors are distributed at random in space but with comparable velocity magnitudes as the true mean velocity this would result in a decrease of the mean velocity as for the u component presented in Table 4. This postulated noise component introduced by the variation of the peak validation ratio is expected to depend on the size of the interrogation area used; larger areas correspond with fewer false positive vectors since the amount of particles comprising the velocity information increases correspondingly. These simple considerations already demonstrate that the turbulent quantities computed from PIV may vary in a large range just by changing the parameter used for the analysis. Thus it is necessary to precisely know the effect of each validation and processing algorithm in advance before treating PIV recordings. This would necessitate more intensive study of the influence of the relevant parameter of the kind we tried to initiate in this article. Consequently, it is recommended that a quality check corresponding to the grid size independence proof mandatory for numerical calculations is developed for PIV calculations. This check could assume the form of a parameter independence proof specific to the PIV setup.

Advantages of PIV and Optical Flow

The main advantage of PIV compared with LDA measurements is the possibility to analyze and visualize instantaneous flow structures to obtain a qualitative and quantitative overview of the vortex and turbulence structures. These instantaneous velocity fields can subsequently be used to compute instantaneous spatial gradients of the velocity and gradient fluctuations. For this reason, PIV allows the detection of vortex structures and the calculation of components for the turbulent kinetic energy balance such as dissipation and production of turbulent kinetic energy. This is still with the limitation, however, that quantitative descriptions must be handled with care as discussed in the previous section.
One instantaneous flow structure from the PIV measurement for the high turbulence case is presented in Figure 5. A part of the von Karman vortex street forming behind the cylinder is shown. A considerable number of vectors (the grey vectors) are rejected by the validation algorithms.

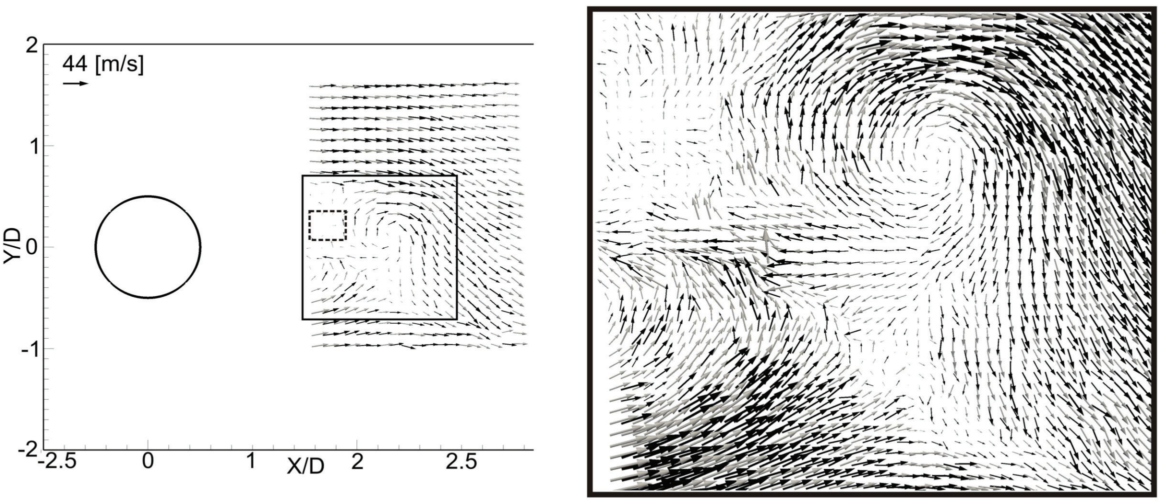


Figure 5: Instantaneous PIV-Measurement, processed with cross correlation.

In comparison to this, the same image analyzed with the optical flow method is shown in Figure 6. It is apparent that the vector density is considerably enhanced as depicted in the lower part of Figure 6, and also that smaller flow structures which are not resolved with the cross-correlation approach become visible. Due to the principle of the optical flow method which forces a homogeneous spatial velocity distribution, the optical flow appears to be very useful for characterization of small scale flow structures from PIV-images.

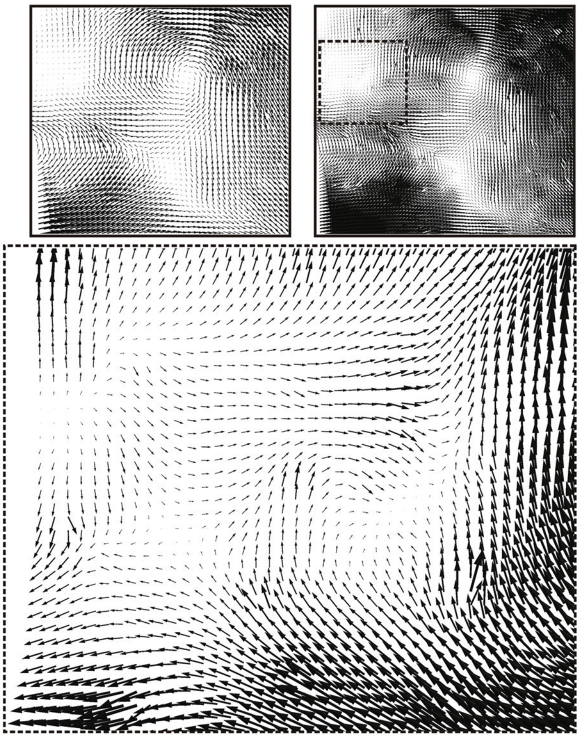


Figure 6: Instantaneous PIV measurement, processed with Optical Flow

Summary and Conclusion

The comparison of PIV with LDA measurements has revealed that reliable mean velocities which deviate less than 1 % from their converged value can be calculated from 2500 samples. Using this minimum number of samples n=2500, the corresponding fluctuating quantities differ by 3% from the converged values. It must be noted, however, that applying different analysis codes or using different user adjustable parameters as the peak ratio validation, even for the converged data using up to 4000 samples, results in differences of the result with the same magnitude when compared with LDA. PIV measurements can give a very good overview of the flow and turbulence structures. Up-to-date processing algorithms such as adaptive correlation and Optical Flow in particular result in very detailed information about flow structures. Regardless, a quantification of turbulence with PIV must still be handled with care as it depends strongly on the choice of the processing parameters.

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Literature

Adrian, R.J. (1984), Scattering particle characteristics and their effect on pulsed laser measurements of fluid flow: speckle velocimetry vs. particle image velocimetry. Appl. Opt. 23: 1690–1691

Corpetti, T., Heitz, D., Arroyo, G., M´emin, E., Santa-Cruz, A. (2006), Fluid experimental flow estimation based on an Optical-flow scheme. Exp. in Fluids 40, 80–97.

Dantec Dynamics (2002), Flow Manager Software and Introduction to PIV Instrumentation, Dantec Dynamics A/S, P.O. Box 121, Tonsbakken 18, DK-2740 Skovlunde, Publication No. 9040U3625

Barker, D.B., M.E. Fourney (1977), Measuring fluid velocities with speckle patterns. Opt. Lett. 1:135–137

Dudderar, T.D., P.G. Simpkins (1977), Laser speckle photography in a fluid medium. Nature 270:45–47

Grousson, R., S. Mallick (1977), Study of flow pattern in a fluid by scattered laser light. Appl Opt 16:2334–2336

Heitz, D.; H´eas, P.; Navaza, V.; Carlier, J.; M´emin, E. (2007), Spatio-temporal correlation-variational approach for robust optical flow estimation, 7th International Symposium on Particle Image Velocimetry - PIV2007 - Roma, 11 - 14 September 2007

Horn, B.K.P and Schunk, B.G.: Determining optical flow. Artificial Intelligence, 17, pp. 185-204, 1981.

Kapulla, R.; Dyck, C.; Witte, M.; Fokken, J.; Leder, A. (2009) Optischer Fluss und Kreuzkorrelation zur Auswertung von Geschwindigkeitsfeldern, In Proceedings der 17. GALA-Fachtagung "Lasermethoden in der Strömungsmesstechnik", 8. - 10. September 2009, Universität Erlangen, Hrsg.: Delgado, A., Rauh, C., Lienhart, H., Ruck, B., Leder, A., Dopheide, D., ISBN 978-3-9805613-5-8, 502 Seiten

Keane, R.D., Adrian, R. J. (1992), Theory of cross-correlation analysis of PIV images, Applied Scientific Research, Vol. 49, 191-215, DOI 10.1007/BF00384623

Nogueira J, Lecuona A and Rodríguez PA (1997) Data validation, false vectors correction and derived magnitudes calculation on PIV data , Measurement Science and Techology, Vol 8, no. 12, pp. 1493-1501

Pickering, C.J.D., N. Halliwell (1984), LSP and PIV: photographic film noise. Appl Opt 23:2961–2969

Raffel, M., Willert, C. E., Kompenhans, J. (1998), Particle Image Velocimetry, a practical guide, Springer Verlag

Ruhnau, P., Kohlberger, T., Nobach, H. and Schnörr, C. (2005), Variational Optical Flow Estimation for Particle Image Velocimetry, Experiments in Fluids, 38:21-32.

Willert, C.E., M. Gharib (1991), Digital particle image velocimetry, Experiments in Fluids 10: 181-193, DOI 10.1007/BF00190388